

Finite-dimensional representations of the queer Lie supergroup $Q(n)$

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The queer Lie superalgebra $\mathfrak{q}(n)$

Definition

The *queer Lie supergroup* $Q(n)$ is the subsupergroup of automorphisms of $\mathbf{C}^{n|n}$ which preserve an odd automorphism Π of $\mathbf{C}^{n|n}$ with the property $\Pi^2 = -1$. The *queer Lie superalgebra* $\mathfrak{q}(n) := \text{Lie}(Q(n))$ is the Lie subsuperalgebra of endomorphisms in $\mathfrak{gl}(n|n)$ which super commute with Π . In coordinates,

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n|n) \right\}, \quad \text{for } \Pi = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Definition (Important subsuperalgebras of $\mathfrak{q}(n)$)

- The *Borel subsuperalgebra* \mathfrak{b} is defined by requiring A, B to be upper triangular.
- The *nilpotent subsuperalgebra* \mathfrak{n} is defined by requiring A, B to be strictly upper triangular.
- The *Cartan subsuperalgebra* \mathfrak{h} is defined by requiring A, B to be diagonal.

In general, the Cartan is defined as a maximal self-normalizing nilpotent subsuperalgebra. $\mathfrak{q}(n)$ is the only “classical” superalgebra in Kac’s classification for which \mathfrak{h} is **not abelian**.

Clifford supermodules = \mathfrak{h} -supermodules

To each weight $\lambda \in \mathfrak{h}_0^*$, we associate a simple \mathfrak{h} -supermodule as follows:

- 1 Define a symmetric bilinear pair $F_\lambda : \mathfrak{h}_{\bar{1}} \times \mathfrak{h}_{\bar{1}} \rightarrow \mathbf{C}$, $(u, v) \mapsto \lambda([u, v])$
- 2 Denote the induced non-degenerate Clifford algebra by $\text{Cliff}(\lambda) := (E_\lambda, F_\lambda)$, where $E_\lambda = F_\lambda / \text{Ker}(F_\lambda)$.
- 3 If $\dim(E_\lambda)$ is even, there are 2 simple $\text{Cliff}(\lambda)$ -supermodules, denoted $v(\lambda)$ and $\Pi v(\lambda)$. If $\dim(E_\lambda)$ is odd, there is unique simple $\text{Cliff}(\lambda)$ -supermodule, denoted $v(\lambda)$.

Theorem

The supermodule $v(\lambda)$ has an \mathfrak{h} -action where \mathfrak{h}_0 acts by λ and $\mathfrak{h}_{\bar{1}}$ acts by $\text{Cliff}(\lambda)$. This construction produces all irreducible finite-dimensional \mathfrak{h} -supermodules, and the category $\mathfrak{h}\text{-mod}$ is semisimple.



Classification of irreducible $Q(n)$ -modules.

Definition

- Verma module $M(\lambda) := U(\mathfrak{q}(n)) \otimes_{U(\mathfrak{b})} v(\lambda)$ and the unique simple quotient is denoted $L(\lambda)$.
- The lattice of *dominant integral weights* is

$$\Lambda^+ := \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}_0^* : \lambda_i - \lambda_{i+1} \in \mathbf{Z}_{\geq 0}, \lambda_i = \lambda_j \Rightarrow \lambda_i = \lambda_j = 0\}.$$

Theorem (Kac '77)

The $\mathfrak{q}(n)$ -supermodule $L(\lambda)$ is finite-dimensional if and only if $\lambda \in \Lambda^+$.

Theorem (Penkov-Serganova '97)

For weights λ which are generic (i.e. $\lambda_i \gg \lambda_{i+1}$) and typical (i.e. $\lambda_i + \lambda_j \neq 0 \forall i, j$), the character formula is

$$ch(L(\lambda)) = \frac{\dim v(\lambda)}{D} \sum_{w \in S_n} \text{sgn}(w) \cdot w[e^{\lambda + \rho_0} \cdot \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha})]$$

The category of $Q(n)$ -supermodules

Definition

Let $Q(n)\text{-mod}$ denote the category of finite-dimensional $Q(n)$ -supermodules with parity-preserving morphisms.

The category $G\text{-mod}$ is equivalent to $(\mathfrak{g}, G_{\bar{0}})\text{-mod}$, i.e \mathfrak{g} -supermodules such that

- ① Morphisms preserve the $\mathbf{Z}/2$ -grading
- ② $\mathfrak{g}_{\bar{0}}$ acts semisimply.

There is a block-decomposition parameterized by central characters

$$Q(n)\text{-mod} = \bigoplus_{\chi: Z(\mathfrak{g}) \rightarrow \mathbf{C}} (Q(n)\text{-mod})_{\chi}$$

Theorem (Sergeev '83)

For $\lambda, \mu \in \mathfrak{h}_{\bar{0}}^*$, $\chi_{\lambda} = \chi_{\mu}$ if and only if $\varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_n} = \varepsilon_{\mu_1} + \cdots + \varepsilon_{\mu_n}$, where $\varepsilon_{\lambda_i} = -\varepsilon_{-\lambda_i}$.

Block equivalences

The most important tool in studying blocks is *translation functors*

$$T_{\lambda, \mu} : (Q(n) - \text{mod})_{\lambda} \rightarrow (Q(n) - \text{mod})_{\mu}, \quad M \mapsto \text{pr}_{\mu}(M \otimes L(1, 0, \dots, 0)).$$

Theorem (Ser-ICM '14)

Let $\lambda \in \Lambda^+ \cap \mathbf{Z}^n$. Then $(Q(n) - \text{mod})_{\lambda}$ is block equivalent to $(Q(n) - \text{mod})_{\tilde{\lambda}}$, where

- (a) $\tilde{\lambda} = (1, 0, \dots, 0)$ for $q(k)$, $k \leq n$. This is the standard block.
- (b) $\tilde{\lambda} = (0, \dots, 0)$ for $q(k)$, $k \leq n$. This is the principal block.

The integer k is called the *atypicality* of the weight λ .

Recently, Brundan gave a similar block equivalence theorem for half-integral weights based on *atypicality*.



Induction on flag supervarieties

Definition

Let $P \subset G$ be any parabolic subsupergroup. The *flag supervariety* G/P exists in the category of superschemes and coincides with Manin's flag superspace, denoted $F_{\text{Spec} \mathbf{C}}(a_1 | a_1, \dots, a_k | a_k; \mathbf{C}^{n|n})$.

Definition

For any P -module V , let $\mathcal{V} = G \times_P V$ denote the vector bundle over G/P . Define the *geometric induction functor* by

$$\Gamma_i : P\text{-mod} \rightarrow G\text{-mod}, \quad V \mapsto H^i(G/P, \mathcal{V}^*)^*$$

There is an equivalent algebraic definition of Γ_i using Zuckerman's functor:

$$H^0(G/P, \mathcal{V}) \cong \Gamma_{\mathfrak{g}_0}(\text{Hom}_{U_P}(U\mathfrak{g}, V))$$

So, $\Gamma_0(V)$ is just the maximal finite-dimensional quotient of $U\mathfrak{g} \otimes_{U_P} V$.



(virtual) BGG Reciprocity

Lemma (Typical Lemma)

Suppose $\lambda \in \Lambda^+$ is \mathfrak{p} -typical. Then

$$\Gamma_i(G/P, v_{\mathfrak{p}}(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0. \\ 0 & \text{if } i > 0. \end{cases}$$

Definition

Let $\lambda \in \Lambda$ be any weight. Define the *Euler characteristic*

$$\mathcal{E}(\lambda) := \sum_{\mu \in \Lambda^+} \sum_{i=0}^{\dim(G/B)_{\bar{0}}} (-1)^i [\Gamma_i(G/B, v(\lambda)) : L(\mu)] \cdot [L(\mu)] \in \mathcal{K}^{\Pi}(Q(n) - \text{mod}).$$

The typical lemma says that generically, $\text{ch}(\mathcal{E}(\lambda)) = \text{ch}(L(\lambda))$.



(virtual) BGG Reciprocity

Theorem (G. Serganova, '20)

Let $\mu \in \Lambda^+$, $b_{\mu,\lambda} \in \mathbf{Z}$ be coefficients occurring in

$$\mathcal{E}(\mu) = \sum_{\lambda \in \Lambda^+} b_{\mu,\lambda} [L(\lambda)].$$

Then there exists $a_{\lambda,\mu} \in \mathbf{Z}$ such that for $\lambda \in \Lambda^+$,

$$[P(\lambda)] = \sum_{\mu \in \Lambda_0^+} a_{\lambda,\mu} \mathcal{E}(\mu).$$

The coefficients $a_{\lambda,\mu} = 2^{t(\mu)-t(\lambda)} \gamma_{\mu} b_{\mu,\lambda}$, where $t(\lambda)$ is either 0 or 1 and γ_{μ} is either 1 or 2.

We make 2 remarks:

- 1 Our proof generalizes Gruson-Serganova which works whenever $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$.
- 2 Using Penkov-Serganova character formula for $\mathfrak{q}(n)$, we can compute all multiplicities $[P(\lambda) : L(\mu)] = \sum_{\nu \in \Lambda_0^+} a_{\lambda\nu} b_{\nu\mu}$.



Standard block for $\mathfrak{q}(n)$

Theorem (G. Serganova '20)

Let λ, μ be weights in the $\mathfrak{q}(n)$ standard block. Then for $t \gg 0$,

$$\begin{aligned} \mathrm{Ext}_{\mathfrak{q}(n)}^1(L(\lambda), L(\mu)) &= \mathrm{Ext}_{\mathfrak{q}(n)}^1(L(t, \tilde{\lambda}), L(t, \tilde{\mu})) \\ &= \mathrm{Ext}_{\mathfrak{q}(1) \oplus \mathfrak{q}(n-1)}^1(L(t) \boxtimes L(\tilde{\lambda}), L(t) \boxtimes L(\tilde{\mu})) \\ &= \mathrm{Ext}_{\mathfrak{q}(n-1)}^1(L(\tilde{\lambda}), L(\tilde{\mu})) \oplus \mathrm{Ext}_{\mathfrak{q}(n-1)}^1(L(\tilde{\lambda}), \Pi L(\tilde{\mu})), \end{aligned}$$

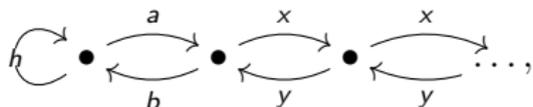
where $\tilde{\lambda}, \tilde{\mu}$ are weights in the $\mathfrak{q}(n-1)$ principal block.



Standard block for $q(3)$ Ext-quiver

Corollary

The Ext-quiver for the $q(3)$ standard block $\lambda = (1, 0, 0)$ is



where vertices are labeled $L(1, 0, 0)$, $L(2, 1, -2)$, $L(3, 1, -3)$, ... with relations

$$\begin{aligned} x^2 = y^2 = 0, & \quad xa = by = ab = 0, \\ h^2 = 0, & \quad xy = yx, \quad bah = hba. \end{aligned}$$

Corollary

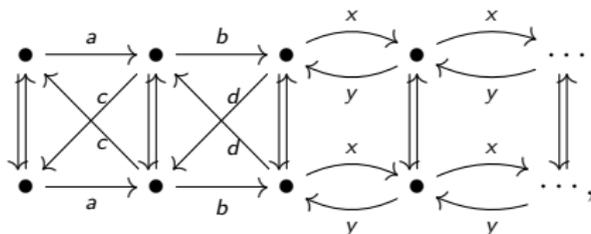
The $q(3)$ standard block is tame, but not a highest-weight category.



Principal block for $q(3)$

Theorem

The Ext-quiver for the $q(3)$ principal block $\lambda = (0, 0, 0)$ is



where vertices are labeled $L(1, 0, -1)$, $L(0)$, $L(2, 0, -2)$, $L(3, 0, -3)$, ... in top and $\Pi L(1, 0, -1)$, $\Pi L(0, 0, 0)$, $\Pi L(2, 0, -2)$, $\Pi L(3, 0, -3)$, ... in bottom. The relations are:

$$\begin{aligned} x^2 = y^2 = 0, \quad xb = dy = bd = ca = 0, \\ xy = yx, \quad yx = bacd, \quad dbac = acdb, \\ \theta^2 = 0, \quad \theta\gamma = \gamma\theta \quad \text{for } \gamma \in \{a, b, c, d, x, y\}. \end{aligned}$$

Corollary

The $q(3)$ principal block is wild and not a highest-weight category.

References I

-  Grantcharov N., Serganova V., Extension Quiver for Lie Superalgebra $q(3)$, *SIGMA* **16** (2020).
-  Gruson C., Serganova V., Bernstein–Gelfand–Gelfand reciprocity and indecomposable projective modules for classical algebraic supergroups, *Mosc. Math. J.* **13** (2013), 281–313, arXiv:1111.6959.
-  Manin Yu.I., Gauge field theory and complex geometry, *Grundlehren der Mathematischen Wissenschaften*, Vol. 289, Springer-Verlag, Berlin, 1997.
-  Penkov I., Serganova V., Characters of finite-dimensional irreducible $q(n)$ -modules, *Lett. Math. Phys.* **40** (1997), 147–158.
-  Serganova V., Finite dimensional representations of algebraic supergroups, in Proceedings of the International Congress of Mathematicians – Seoul 2014, Vol. 1, Kyung Moon Sa, Seoul, 2014, 603–632.

